# A Course
# on
# Convex Geometry

Daniel Hug, Wolfgang Weil
University of Karlsruhe

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# Preface

The following notes were written before and during the course on *Convex Geometry* which was held at the University of Karlsruhe in the winter term 2002/2003. Although this was the first course on this topic which was given in English, the material presented was based on previous courses in German which have been given several times, mostly in summer terms. In comparison with these previous courses, the standard program was complemented by sections on surface area measures and projection functions as well as by a short chapter on integral geometric formulas. The idea here was to lay the basis for later courses on *Stochastic Geometry*, *Integral Geometry* etc., which usually follow in a subsequent term.

The exercises at the end of each section contain all the weekly problems which were handed out during the course and discussed in the weakly exercise session. Moreover, I have included a few additional exercises (some of which are more difficult) and even some hard or even unsolved problems. The list of exercises and problems is far from being complete, in fact the number decreases in the later sections due to the lack of time while preparing these notes.

I thank Matthias Heveling and Markus Kiderlen for reading the manuscript and giving hints for corrections and improvements.



Karlsruhe, February 2003 Wolfgang Weil

During repetitions of the course in 2003/2004 and 2005/2006 a number of misprints and small errors have been detected. They are corrected in the current version. Also, additional material and further exercises have been added.

Karlsruhe, October 2007 Wolfgang Weil

During the courses in 2008/2009 (by D. Hug) and 2009/2010 (by W. Weil) these lecture notes have been revised and extended again. Also, some pictures have been included.

Karlsruhe, October 2009 Daniel Hug and Wolfgang Weil

# Contents

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# Introduction

Convexity is an elementary property of a set in a real (or complex) vector space $V$. A set $A \subset V$ is convex if it contains all the segments joining any two points of $A$, i.e. if $x, y \in A$ and $\alpha \in [0,1]$ implies that $\alpha x + (1-\alpha)y \in A$. This simple algebraic property has surprisingly many and far-reaching consequences of geometric nature, but it also has topological consequences (if $V$ carries a compatible topology) as well as analytical ones (if the notion of convexity is extended to real functions via their graphs). The interplay between convex sets and functions turns out to be particularly fruitful. Results on convex sets and functions play a central role in many mathematical fields, in particular in functional analysis, in optimization theory and in stochastic geometry.

During this course, we shall concentrate on convex sets in $\mathbb{R}^n$ as the prototype of a finite dimensional real vector space. In infinite dimensional spaces often other methods have to be used and different types of problems occur. Here, we concentrate on the classical part of convexity. Starting with convex sets and their basic properties (in Chapter 1), we briefly discuss convex functions (in Chapter 2), and then come (in Chapter 3) to the theory of convex bodies (compact convex sets). Our goal here is to present the essential parts of the Brunn-Minkowski theory (mixed volumes, quermassintegrals, Minkowski inequalities, in particular the isoperimetric inequality) as well as some more special topics (surface area measures, projection functions). In the last chapter, we will shortly discuss selected basic formulas from integral geometry. If time permits we will discuss symmetrization of convex sets and functions in an additional chapter.



The course starts rather elementary. Apart from a good knowledge of linear algebra (and, in Chapter 2, analysis) no deeper knowledge of other fields is required. Later we will occasionally use results from functional analysis, in some parts, we require some familiarity with topological notions and, more importantly, we use some concepts and results from measure theory.

# Preliminaries and notations

Throughout the course we work in $n$-dimensional Euclidean space $\mathbb{R}^n$. Elements of $\mathbb{R}^n$ are denoted by lower case letters like $x, y, \ldots, a, b, \ldots$, scalars by greek letters $\alpha, \beta, \ldots$ and (real) functions by $f, g, \ldots$ We identify the vector space structure and the affine structure of $\mathbb{R}^n$, i.e. we do not distinguish between vectors and points. The coordinates of a point $x \in \mathbb{R}^n$ are used only occasionally, therefore we indicate them as $x = (x^{(1)}, \ldots, x^{(n)})$. We equip $\mathbb{R}^n$ with its usual topology generated by the standard scalar product

$$\langle x, y \rangle := x^{(1)}y^{(1)} + \cdots + x^{(n)}y^{(n)}, \quad x, y \in \mathbb{R}^n,$$

and the corresponding Euclidean norm

$$\|x\| := ((x^{(1)})^2 + \cdots + (x^{(n)})^2)^{1/2}, \quad x \in \mathbb{R}^n.$$

By $B^n$ we denote the unit ball,

$$B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\},$$



and by

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$$

the unit sphere. Sometimes, we also make use of the Euclidean metric $d(x, y) := \|x - y\|$, $x, y \in \mathbb{R}^n$. Sometimes it is convenient to write $\frac{x}{\alpha}$ instead of $\frac{1}{\alpha}x$, for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Convex sets in $\mathbb{R}^1$ are not very exciting (they are open, closed or half-open, bounded or unbounded intervalls), usually results on convex sets are only interesting for $n \geq 2$. In some situations, results only make sense, if $n \geq 2$, although we shall not emphasize this in all cases. As a rule, $A, B, \ldots$ denote general (convex or nonconvex) sets, $K, L, \ldots$ will be used for compact convex sets (convex bodies) and $P, Q, \ldots$ for (convex) polytopes.

A number of notations will be used frequently, without further explanations:

| | |
|---|---|
| $\operatorname{lin} A$ | linear hull of $A$ |
| $\operatorname{aff} A$ | affine hull of $A$ |
| $\dim A$ | dimension of $A$ (= dimension of $\operatorname{aff} A$) |
| $\operatorname{int} A$ | interior of $A$ |
| $\operatorname{relint} A$ | relative interior of $A$ (interior w.r.t. $\operatorname{aff} A$) |
| $\operatorname{cl} A$ | closure of $A$ |
| $\operatorname{bd} A$ | boundary of $A$ |
| $\operatorname{relbd} A$ | relative boundary of $A$ |

If $f$ is a function on $\mathbb{R}^n$ with values in $\mathbb{R}$ or in the extended real line $[-\infty, \infty]$ and if $A$ is a subset of the latter, we frequently abbreviate the set $\{x \in \mathbb{R}^n : f(x) \in A\}$ by $\{f \in A\}$. Hyperplanes $E \subset \mathbb{R}^n$ are therefore shortly written as $E = \{f = \alpha\}$, where $f$ is a linear form, $f \neq 0$, and $\alpha \in \mathbb{R}$ (note that this representation is not unique). The corresponding closed half-spaces generated by $E$ are then $\{f \geq \alpha\}$ and $\{f \leq \alpha\}$, and the open half-spaces are $\{f > \alpha\}$ and $\{f < \alpha\}$.

The symbol $\subset$ always includes the case of equality. The abbreviation w.l.o.g. means 'without loss of generality' and is used sometimes to reduce the argument to a special case. The logical symbols $\forall$ (for all) and $\exists$ (exists) are occasionally used in formulas. $\square$ denotes the end of a proof. Finally, we write $|A|$ for the cardinality of a set $A$.

Each section is complemented by a number of exercises. Some are very easy, but most require a bit of work. Those which are more challenging than it appears from the first look are marked by $*$. Occasionally, problems have been included which are either very difficult to solve or even unsolved up to now. They are indicated by P.