Preface

The following notes were written before and during the course on *Convex Geometry* which was held at the University of Karlsruhe in the winter term 2002/2003. Although this was the first course on this topic which was given in English, the material presented was based on previous courses in German which have been given several times, mostly in summer terms. In comparison with these previous courses, the standard program was complemented by sections on surface area measures and projection functions as well as by a short chapter on integral geometric formulas. The idea here was to lay the basis for later courses on *Stochastic Geometry, Integral Geometry* etc., which usually follow in a subsequent term.

The exercises at the end of each section contain all the weekly problems which were handed out during the course and discussed in the weakly exercise session. Moreover, I have included a few additional exercises (some of which are more difficult) and even some hard or even unsolved problems. The list of exercises and problems is far from being complete, in fact the number decreases in the later sections due to the lack of time while preparing these notes.

I thank Matthias Henning and Markus Kiderlen for reading the manuscript and giving hints for corrections and improvements.

Karlsruhe, February 2003 Wolfgang Weil

During repetitions of the course in 2003/2004 and 2005/2006 a number of misprints and small errors have been detected. They are corrected in the current version. Also, additional material and further exercises have been added.

Karlsruhe, October 2007 Wolfgang Weil

During the courses in 2008/2009 (by D. Hug) and 2009/2010 (by W. Weil) these lecture notes have been revised and extended again. Also, some pictures have been included.

Karlsruhe, October 2009 Daniel Hug and Wolfgang Weil

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Introduction

Convexity is an elementary property of a set in a real (or complex) vector space V. A set $A \subset V$ is convex if it contains all the segments joining any two points of A, i.e. if $x, y \in A$ and $\alpha \in [0,1]$ implies that $\alpha x + (1-\alpha)y \in A$. This simple algebraic property has surprisingly many and far-reaching consequences of geometric nature, but it also has topological consequences (if V carries a compatible topology) as well as analytical ones (if the notion of convexity is extended to real functions via their graphs). The interplay between convex sets and functions turns out to be particularly fruitful. Results on convex sets and functions play a central role in many mathematical fields, in particular in functional analysis, in optimization theory and in stochastic geometry.

During this course, we shall concentrate on convex sets in $\mathbb{R}^n$ as the prototype of a finite dimensional real vector space. In infinite dimensional spaces often other methods have to be used and different types of problems occur. Here, we concentrate on the classical part of convexity. Starting with convex sets and their basic properties (in Chapter 1), we briefly discuss convex functions (in Chapter 2) and then come to the theory of convex bodies (compact convex sets). Our goal here is to present the essential parts of the Brunn-Minkowski theory (mixed volumes, quermassintegrals, Minkowski inequalities, in particular the isoperimetric inequality) as well as some more special topics (surface area measures, projection functions). In the last chapter, we will shortly discuss selected basic formulas from integral geometry. If time permits we will discuss symmetrization of convex sets and functions in an additional chapter.

The course starts rather elementary. Apart from a good knowledge of linear algebra (and, in Chapter 2, analysis) no deeper knowledge of other fields is required. Later we will occasionally use results from functional analysis, in some parts, we require some familiarity with topological notions and, more importantly, we use some concepts and results from measure theory.