Group Representation Theory for Physicists

2nd Edition

Jin-Quan Chen, Jialun Ping & Fan Wang

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GROUP REPRESENTATION THEORY FOR PHYSICISTS, 2nd Edition

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Foreword to the first edition

A new generation of books on group theory for physicists has appeared over the last ten years. Many of them deal only with elementary particle physics or with condensed matter physics. This volume by Prof. Jin–Quan Chen is a serious attempt to cover a broad range of applications of group theory to physics. It begins with an introduction to the elements of group theory and the theory of representations. Representations of finite groups and character theory are carefully treated and applied in later chapters to point groups and space groups, where thorough and practical information is given about molecular and crystal groups.

The permutation group is discussed in detail, and its use in finding the irreducible representations of unitary groups is carefully and completely covered. Lie groups and Lie algebras are treated sufficiently to enable their use in elementary particle physics, with a thorough presentation of Dynkin diagrams and the reduction of products.

For spectroscopy and, in particular, nuclear spectroscopy, this is the first up–to–date and thorough treatment of the calculation of isoscalar factors and coefficients of fractional parentage. There are extensive tables and indications of computational methods.

With his collaborators in Nanjing and Philadelphia, Chen has contributed extensively to new developments over the past decade. These researches are incorporated into the present book, of which a preliminary version was published a few years ago in the People's Republic of China.

It has been a pleasure for me to talk with Chen many times about group theory and a special pleasure to welcome the appearance of this book.

Morton Hamermesh
University of Minnesota

Preface to the first edition

Conscious of the frustration we experienced as we tried to learn group theory, and apply it to problems in physics, Fan Wang, Mei Juan Gao and I endeavored in 1974 to carry out a systematic reform of standard, or traditional group representation theory. Our aim was to establish a new approach to group representations in accordance with the concept and method used in quantum mechanics, so that it would be much more accessible to physicists and chemists.

The breakthrough came in late 1974. A new approach to group representation gradually came into being and with it a new method, the eigenfunction method (EFM), emerged for finding primitive characters, irreducible bases, Clebsch–Gordan (CG) coefficients, etc. From 1976–1978, the EFM was translated into codes and for the first time the CG coefficients and the outer-product reduction coefficients for the permutation group S_2–S_6 were computed by the EFM. The EFM was extended in 1979 to the computation of the CG coefficients, Racah coefficients, isoscalar factors, subduction coefficients of unitary groups. From 1981–1987, the new approach was applied to space groups and graded unitary groups and established several codes for space groups, SU(n) CG coefficients, SU(mn) $\supset$ SU(m) $\times$ SU(n) and SU($m+n$) $\supset$ SU(m)$\times$ SU(n) coefficients of fractional parentage, and so on.

The new approach has been introduced at various universities and institutions, such as the City College of the City University of New York, the University of Chicago, Clarkson, Dalhousie, Drexel, Duke, Hong Kong, Minnesota, Maryland, Montreal, Singapore, Yale, and Los Alamos National Laboratory, as well as at the IXth and XIVth International Colloquium on Group Theoretical Method in Physics held in Mexico 1980, and Seoul 1985, respectively.

The Chinese book *A New Approach to Group Representation Theory* was written in 1981 and published in 1985 and has since been used as a textbook in Nanjing University for undergraduate and graduate students. Prior to the publication of the Chinese book, the manuscript had been used as a textbook in Nanjing University and Drexel University in Philadelphia.

The present book was based on the Chinese book and the new developments during the period of 1981–1987. All chapters except chapters 4 and 7 have been re-written.

I am indebted to my collaborators Mei-Juan Gao who took part in the writing of Chapter 8 and performed most of the practical computations, Guang-Qun Ma who was involved in the writing of Chapter 10, and it is he who first attempted to use the EFM to treat the space group representation, Fan Wang and Xuan-Gen Chen for their invaluable suggestions and criticism. I am grateful to Professors Xiao-Qian Zhou, Shi-Shu Wu and Duan Feng for their constant encouragement and patient reading of the manuscript. Thanks are due also to Professors Tan Lu, Fu-Cho Pu, Qin-Yue Qu, Hong-Zhou Sun, Rui-Bao Tao, Cheng-Li Wu, Rong-Jue Wei, Xi-De Xie and Guang-Xiang Xu for their enthusiasm in this work and support. I am grateful to many students who have offered helpful criticism and raised challenging questions.

Mr. Pei-Ning Wang typed the initial draft of Chapters 4–9, and my son, Bing-Qing wrote several software codes for facilitating the typing and proof-reading of the manuscript. Without their help it would be inconceivable that the re-writing could have been accomplished within

half a year.

Special mention must be made of Professors B. Bayman, M. Hamermesh and K.T. Hecht for their constant interest and many illuminating discussions, and D.H. Feng, who gave me the opportunity and encouragement to teach a course on group theory at Drexel University.

Last but not least I would like to thank Professors L. C. Biedenharn, J. Birman, U. Fano, R. Gilmore, J. Ginocchio, J. J. Griffin, F. Iachello, J. Paldus, J. Patera and K. K. Phua for their stimulating discussions and the hospitality extended to me during my visits to their institutions.

Nanjing University Jin–Quan Chen
March, 1987

Preface to the second edition

Ten years have passed since the publication of the first edition of this book in which a complete set of commuting operators (CSCO) approach to the group representation theory was proposed. During this period, several remarkable advances of relevance have been made. A common feature of these new developments is that they give algebraic expressions for some useful coefficients of the following groups:

1. Quantum groups $U_q(n)$ (Pan & Chen 1993).

2. Orthogonal groups O_n and symplectic groups $Sp(2n)$ (Pan & Dai, 1996, Pan, Dong & Draayer, 1997, 1998).

3. Unitary groups and permutation groups with irreducible representations involving two columns (Li & Paldus, 1990, 1993).

4. Point groups (Chen & Fan 1998, Fan & Chen, 1999, Fan, Chen & Draayer 1999).

The first three will not be covered in this book but will be mentioned in appropriate places in the book, while some new results for point groups will be introduced in Chapter 8.

In the 1989 edition of the book, the projection operator method, the most commonly used method in standard textbooks on group theory, was essentially ignored. We mentioned only the main steps and discussed its serious shortcomings. Instead we developed the eigenfunction method (EFM) and urged its use. Ten years later, the wheel has gone full circle, and it is time to re-assess the power of the projection operator method. With the discovery of algebraic expressions for the projection operators, the projection operator method becomes extremely simple and powerful for point groups. An interesting point though is that only with the help of the EFM are these algebraic expressions obtainable. So we have a good example of negating a negation. The projection operator returns, albeit in a new form.

In this edition the following sections are new:

Sec. 3.14. Irreducible basis vectors in non-orthogonal reducible basis.

Sec. 5.9. The groups U_{2j+1}, SO_{2l+1} and Sp_{2j+1}.

Sec. 7.20. The generalized quantized expressions for the coefficients of fractional parentage.

Sec. 9.8. The Groups Sp_N, SO_N and the pairing interaction.

Sec. 9.2. Nuclear shell model: Multi-shell.

Most parts of Chapter 8 on the point groups have been rewritten. The new contents include a preliminary introduction of the algebraic solutions and a detailed discussion of the double-valued representations of point groups.

The assistance of Dr. L. McAven and Prof. C. W. Wong has been invaluable in preparing this edition, and I am very grateful to them both. Dr. McAven helped with the drawing of all of the figures, checked the text and formulas, and assisted with layout issues. Prof. C. W. Wong carefully read the book, in particular improving upon the exposition of Sec. 3.9.3. Discussions with Drs. Peng-Dong Fan, S. E. Bernades and Hong-Zhou Sun have also been useful.

Nanjing University
Dec., 1999

Jin–Quan Chen

Addendum

As mentioned by Prof. J. Q. Chen in the preface of the first edition, we started to reform the group representation theory in accordance with the concept and method of quantum mechanics in 1974. Because both of us kept an idea that due to the group theory was solely developed by mathematicians, so it does not match the quantum physics well and it should be possible to have a group representation theory which matches quantum physics as harmonious as the relation between calculus and the classical physics. Such a reform was realized in 1974-1980. Since 1977 I was interested in the hadron physics based on quantum chromodynamics inspired models and gradually shifted to that field. All the further developments after 1981 were by Chen himself and later with collaboration with Jia-Lun Ping and Feng Pan except the application of the new representation theory in the quark model calculations.

Prof. J. Q. Chen passed away untimely on May 14, 2000. This second edition is almost finished by Chen himself. As a lifelong colleague of Prof. Chen, I am duty-bound to make it published. The unfinished work is mainly done by Prof. Jia-Lun Ping.

We offer this book to the community interested in the application of group theory as a monument of Prof. Chen. It includes every contributions of Prof.Chen to the group theory and its applications except the graded Lie group and quantum group which Prof. Feng Pan will explain it in the memorial volume "The Intellectual Path of J. Q. Chen: A Memorial".

Fan Wang
February, 2002 at Nanjing

Contents

Chapter 5 Lie Groups

Appendix

Contents of Some Important Tables

Glossary

1. Group

E	Euclidean group
G	a group
$\mathcal{G}$	a representation group
$\overline{G}$	Intrinsic group of G
$\mathbf{G}$	a space group, or an abstract group
$\mathbf{G}^{\dagger}$	double group or double space group
$\mathbf{g}$	a point group
$\mathbf{G}_0$	isogonal point group of the space group $\mathbf{G}$
$\mathbf{G}_0(\mathbf{k})$	little co–group, (10-57a)
$G(\mathbf{k})$	little group, (10-58)
$\mathbf{G}_{\mathbf{k}}, \mathbf{G}'_{\mathbf{k}}$	representation groups, (10-63c), (10-70)
G_s	a subgroup of G
$G(s) = G(1) \supset G(2) \supset \ldots$	a canonical subgroup chain of G
$\overline{G}(s) = \overline{G}(1) \supset \overline{G}(2) \supset \ldots$	a canonical subgroup chain of $\overline{G}$
$G(s')$	a non–canonical subgroup chain
$\mathbf{P}$	holosymmetric point group of a crystal system, (10-17)
$\mathbf{P}(\mathbf{k})$	symmetric group of $\mathbf{k}$, (10-50)
$S_n(\mathcal{S}_n)$	permutation group in coordinate (state) space
$S_n(\omega)$	a permutation group in the indices (ω)
$\mathbf{T}$	translation group, or lattice group
$U_n, SU_n(\mathcal{U}_n, \mathcal{SU}_n)$	unitary group in coordinate (state) space

2. Group elements

$a, R_a, R(a), R, S, T, \ldots$	group elements
α, β, γ	group elements of a point group
p	permutation operator in coordinate space, (3-108)
$\wp$	permutation operator in state space, (3-107)
$Q = \binom{\omega_0}{\omega}$	order–preserving permutation, (4-131)
$R_i = \{\gamma_i \vert \mathbf{V}(\gamma_i)\}' = \exp[i\mathbf{k} \cdot \mathbf{V}(\gamma_i)]\{\gamma_i \vert \mathbf{V}(\gamma_i)\}$	group element of the rep group $G'_{\mathbf{k}}$, (10-66)

3. Spaces and basis vectors

$L, \mathcal{L}$	any rep space
L_g	group space of a group G, or a rep group G
$L_c(L_n)$	class space of a group G (a rep group $G'_{\mathbf{k}}$)
L_ν	eigenspace of the CSCO–I of $G, L \subset L_g$
$\mathcal{L}_\nu$	eigenspace of the CSCO–I of $G, \mathcal{L}_\nu \subset \mathcal{L}$
$L_{(\nu)k}$	the k–th irreducible space of G
$L(\mathbf{k})$	group space of the rep group $G'_{\mathbf{k}}$, (10-73b)
$\mathcal{L}_{\mathbf{k}}$	eigenspace of translation operator $\{\varepsilon \vert \mathbf{R}_n\}$, (10-54b)
$\mathcal{L}(*\mathbf{k})$	representation space of a space group $\mathbf{G}$, (10-55c)
$\vert\omega_0\rangle = \vert i_1 i_2 \ldots i_n\rangle$	normal order state, (3-106)
$\vert\omega\rangle = \vert i_1 i_2 \ldots i_n\rangle$	normal order state (with repeated state labels)

[a] The equation number gives the place where the symbol appears for the first time.

4. Complete set of commuting operators

$C(C(i))$	CSCO–I, or CSCO of $G(G(i))$
$C(s) = (C(1), C(2), \ldots)$	complete set of commuting operators of $G(s)$
$\overline{C}(s) = (\overline{C}(1), \overline{C}(2), \ldots)$	complete set of commuting operators of $\overline{G}(s)$
$C(s')$	CSCO of a non–canonical subgroup chain $G(s')$
$M = (C, C(s))$	CSCO–II of G
$K = (C, C(s), \overline{C}(s))$	CSCO–III of G
$C(n)$	CSCO–I of S_n, or the two–cycle class operator of S_n
$\mathcal{C}(n)$	CSCO–I of the state permutation group $\mathcal{S}_n$

5. Irreducible basis and matrices

$\psi_m^{(\nu)}$	$G \supset G(s)$ irreducible basis, (3-77b)
$\psi_m^{(\nu)k}, P_m^{(\nu)k}$	$G \supset G(s)$ and $\overline{G} \supset \overline{G}(s)$ irreducible basis
$D_{m\overline{m}}^{(\nu)}(R) = \langle \psi_m^\nu \vert R \vert \psi_{\overline{m}}^\nu \rangle,$	irreducible matrix of the element R, (2-95)
$\vert Y_m^{(\nu)} \rangle$	Yamanouchi basis, (4-11c)
$\vert Y_m^{(\nu)}(\omega) \rangle$	Yamanouchi basis of $S_n(\omega)$, (4-132)
$\left\vert \begin{smallmatrix} [\nu] \\ W \end{smallmatrix} \right\rangle, \left\vert \begin{pmatrix} [\nu] \\ (m) \end{pmatrix} \right\rangle$	Gel'fand basis for a unitary group, (4-84)
$\left\vert [\nu], \begin{smallmatrix} \tau[\nu_1][\nu_2] \\ m_1 m_2 \end{smallmatrix} \right\rangle$	$S_n \supset S_{n_1} \otimes S_{n_2}$ basis, (4-165b)
$\left\vert [\nu], \begin{smallmatrix} \tau[\nu_1][\nu_2] \\ W_1 W_2 \end{smallmatrix} \right\rangle$	$SU_{mn} \supset SU_m \times SU_n$, (7-106) $SU_{m+n} \supset SU_m \otimes SU_n$ basis, (7-141)
$\psi_{\mathbf{k}} = \exp[i(\mathbf{k} + \mathbf{K}_m) \cdot \mathbf{r}]$	irreducible basis of the lattice group

6. Coefficients

$C_{\nu_1 m_1, \nu_2 m_2}^{(\nu)\tau, m},$	CG coefficients, (3-282)
$C_{\nu_1 m_1, \nu_2 m_2, \omega}^{(\nu)\tau, m},$	induction coefficients of S_n, (4-140)
$\left[\begin{smallmatrix} \nu \mathbf{k} \nu' \mathbf{k}' \\ \sigma a \sigma' a' \end{smallmatrix} \middle\vert \begin{smallmatrix} \nu'' \mathbf{k}'' \theta \\ \sigma'' a'' \end{smallmatrix} \right],$	CG coefficients for space groups, (10-138a)
$C_{\nu_1 \beta_1 \Lambda_1, \nu_2 \beta_2 \Lambda_2}^{(\nu)\tau, \beta\Lambda},$	$G \supset G_1$ isoscalar factor (ISF), (3-302)
$C_{\sigma\sigma', \mu\mu'}^{[\nu]\beta, [\nu']\beta'},$	$S_n \supset S_{n-1}$ ISF, (4-188)
$C_{\nu_s \nu_s', \nu_t \nu_t'}^{[\nu]\theta, [\nu']\theta'} = \langle \nu_s \nu_t, \nu \vert b^\dagger \vert \nu_s' \nu_t', \nu' \theta' \rangle,$	$S_n \supset S_{n-1}$ ISF, (9-18)
$C_{\sigma\sigma', \mu\mu'}^{[\nu]\beta, [\nu']\beta'},$	$S_n \supset S_{n-1}$ outer–product ISF, (7-252)
$C_{\nu_1 \nu_1', \nu_2 \nu_2}^{[\nu]\beta, [\nu']\beta'} = \langle \nu_1 \nu_2, \nu \vert a_1^\dagger \vert \nu_1' \nu_2, \nu' \beta' \rangle,$	$S_n \supset S_{n-1}$ outer–product ISF,(9-14b)
$C_{[\sigma]\theta\sigma'\sigma'', [\mu]\varphi\mu'\mu''}^{[\nu]\beta, \tau[\nu']\beta'[\nu'']\beta''}$	$S_n \supset S_{n_1} \otimes S_{n_2}$ ISF, (4-200) $S_n \supset S_{n_1} \otimes S_{n_2}$ outer–product ISF, (7-265)
$C_{[\nu']\beta'\sigma'\mu', [\nu'']\beta''\sigma''\mu''}^{[\nu]\tau, \beta[\sigma]\theta[\mu]\varphi},$	$SU_{mn} \supset SU_m \times SU_n$, (7-224a)

	$SU_{m+n} \supset SU_m \otimes SU_n$ ISF, (7-268)
$C^{[\nu]\tau,\alpha L}_{[\nu_1]\alpha_1 L_1,[\nu_2]\alpha_2 L_2}$,	$SU_{2l+1} \supset SO_3$ ISF (or CFP), (7-166)
$C^{[\nu]\tau,\beta ST}_{[\nu_1]\beta_1 S_1 T_1,[\nu_2]\beta_2 S_2 T_2}$,	$SU_4 \supset SU_2 \times SU_2$ ISF (or CFP), (7-176)
$\left\langle \begin{smallmatrix}[\nu]\\ m\end{smallmatrix} \middle\vert [\nu], \begin{smallmatrix}\tau[\nu_1][\nu_2]\\ m_1 m_2\end{smallmatrix} \right\rangle$, or $\langle[\nu]m\vert\tau[\nu_2]m_2\rangle$,	subduction coefficients (SDC) of S_n, (4-168a)
$\left\langle \begin{smallmatrix}[\nu]\\ w\end{smallmatrix} \middle\vert [\nu], \begin{smallmatrix}\tau[\nu_1][\nu_2]\\ w_1 w_2\end{smallmatrix} \right\rangle$,	$SU_{mn} \downarrow (SU_m \times SU_n)$ SDC,(7-115)
	$SU_{m+n} \downarrow (SU_m \otimes SU_n)$ SDC, (7-144)
$P^{(\nu)k}_m = \sqrt{\frac{h_\nu}{g}} \sum_a D^{(\nu)*}_{mk}(R_a) R_a$	normalized projection operator, (3-199)
$P^{(\nu)}_{mk} = \frac{h_\nu}{g} \sum_a D^{(\nu)*}_{mk}(R_a) R_a$	projection operator, (3-222)
$P^{(\nu)} (= \sum_m P^{(\nu)}_{kk})$	projection operator, (3-51), (3-233)
$R^{[\nu]k}(\omega)$,	normalization coefficients, (4-81a)

7. Miscellaneous

$C_i, C(\varphi)$	class operators
$\tilde{C}_{jm} = (-1)^{j+m} C_{j-m}$	time reversal state
$\widetilde{D}$	the transpose of the matrix D
$\varepsilon_i(\nu_1\nu_2\nu)$	phase factor
Λ^ν_m	phase of the Yamanouchi basis vector
$\Lambda^\nu_{\nu'}$	relative phase of Yamanouchi basis vectors, (4-196')
$h_\nu, \vert\nu\vert$	dimension of the irrep ν
$\hat{j} = \sqrt{2j+1}$	
g, $\vert G\vert$, $\vert\mathbf{G}\vert$	order of a group
g_i	number of elements in the class i
N	number of classes
ν_0	the adjoint rep

8. Abbreviations

CAR	Cartesian representation
CG coefficients	Clebsch–Gordan coefficients
CFP	coefficients of fractional parentage
CSCO	complete set of commuting operators
CSW	complete set of weights
DYN	Dynkin representation
EFM	eigenfunction method
FWS	fundamental weight system representation
IDC	induction coefficients of permutation group, or outer–product reduction coefficients
irrep	irreducible representation
IRB	irreducible basis
ISF	isoscalar factor
rep	representation
SALC	symmetry adapted linear combination
SDC	subduction coefficients
SRS	simple roots representation

The main notations for the space group are listed below:

Genealogical relation for the space group and its point groups

$$\begin{gathered}\mathbf{G} \supset \mathbf{G}(\mathbf{k}) \supset \mathbf{G}(s) \supset \mathbf{T}\ , \\ \mathbf{P} \supset \mathbf{G}_0 \supset \mathbf{G}_0(\mathbf{k})\ , \\ \mathbf{P} \supset \mathbf{P}(\mathbf{k})\ , \\ \mathbf{G}_0(\mathbf{k}) = \mathbf{P}(\mathbf{k}) \cap \mathbf{G}_0\end{gathered}$$

Coset decomposition

$$\mathbf{G} = \sum_{\sigma}{}_1^q \oplus \{\beta_\sigma|\mathbf{V}_\sigma\}\mathbf{G}(\mathbf{k})\ ,$$

$$\mathbf{G}_0 = \sum_{\sigma}{}_1^q \oplus \beta_\sigma \mathbf{G}_0(\mathbf{k})$$

Table 1. Space-group elements, IRB and irrep.

	$\mathbf{G}$	$\mathbf{G}(\mathbf{k})^{a}$	$\mathbf{T}$	$\mathbf{G}'^{b}_{\mathbf{k}}$	$\mathbf{G}_0(\mathbf{k})^{c}$
group element	$\{\alpha_i\|\mathbf{V}(\alpha_i)+\mathbf{R}_n\}$	$\{\gamma_i\|\ \mathbf{V}(\gamma_i)+\mathbf{R}_n\}$	$\{\varepsilon\|\mathbf{R}_n\}$	$\{\gamma_i\|\mathbf{V}(\gamma_i)\}'$	γ
basis vector	$\psi^{(\nu)}_{\mathbf{k}_\sigma a} = \{\beta_\sigma\|\mathbf{V}_\sigma\}\psi^{(\nu)}_{\mathbf{k},a}$	$\psi^{(\nu)}_{\mathbf{k},a}$	$\psi_{\mathbf{k}} = e^{i(\mathbf{k}+\mathbf{K}_m)\bullet r}$	$\psi^{(\nu)}_{\mathbf{k},a}$	
irrep	$D^{(*\mathbf{k})(\nu)}_{\tau b,\sigma a}$	$D^{(\mathbf{k})(\nu)}_{ba}$	$\exp(-i\mathbf{k}\bullet\mathbf{R}_n)$	$D^{(\mathbf{k})(\nu)}_{ba}$	$\Delta^{(\nu)}_{ba}$
dimension	qh_ν	h_ν	1	h_ν	h_ν

[a]The irrep of $\mathbf{G}(\mathbf{k})$ is $D^{(\mathbf{k})(\nu)}(\{\gamma|\mathbf{c}\}) = \exp(-i\mathbf{k}\bullet\mathbf{c})\Delta^{(\nu)}(\gamma)$.

[b]$\{\gamma_j|\mathbf{V}\ (\gamma_j)\}' = R_j = \exp[i\mathbf{k}\bullet\mathbf{V}\ (\gamma_j)]\ \{\gamma_i|\mathbf{V}(\gamma_i)\}$, $j = 1,2,\ldots,|\ \mathbf{G}_0\ (\mathbf{k})\ |$, are the active elements of the rep group $\mathbf{G}'_{\mathbf{k}}$.

[c]$\Delta^{(\nu)}$ is the projective irrep of the little co-group $\mathbf{G}_0(\mathbf{k})$. In full we have $\Delta^{(\nu)}(\gamma) = D^{(\mathbf{k})(\nu)}(\{\gamma|\mathbf{V}(\gamma)\}')$.

Introduction

Group theory plays a very important role in physics and chemistry, and its importance continues to grow seemingly endlessly. The representation theory of both finite and compact Lie groups is treated extensively in numerous books and articles. However, they basically follow the same fundamental theory. This theory, which we call the standard group representation theory, seems to be perfect from the mathematical point of view. Nevertheless, it is not totally satisfying from a practical or physical point of view. Many sophisticated physicists, who were quite at home in their own fields, seemed to be afraid of group theory and expressed their dissatisfaction with standard representation theory (Sokolov 1956, Salam 1963, Lipkin 1966, Slater 1975). During the 1930's–50's, outrage, disgust, characterizing group theory as a plague or calling it "*Gruppenpest*" were not atypical reactions by physicists to the use of group theory in physics. The leading American physicist J. Slater commented (p. 60 in his autobiography):

"*The authors of the 'Gruppenpest' wrote papers which are incomprehensible to those like me who had not studied group theory, in which they applied these theoretical results to the study of many electron problem. The practical consequences appeared to be negligible, but everyone felt that to be in the mainstream one had to learn group theory. Yet there were no good texts for which one could learn group theory. It was a frustrating experience, worthy the name of a pest. I had what I can only describe as a feeling of outrage at the turn which the subject had taken....*

As soon as this [Slater's] paper became known, it was obvious that a great many other physicists were disgusted as I had been with the group theoretical approach to the problem. As I heard later, there were remarks made such as 'Slater has slain the Gruppenpest'. I believe that no other piece of work I have done was so universally popular."

The Nobel Laureate A. Salam said in his opening speech for "Seminars on Theoretical Physics" held in Trieste in 1962:

"*In 1951, I had the good fortune of listening to Professor Racah's lecture on Lie groups at Princeton. After attending these lectures, I thought, "This is really too hard. I cannot learn all this ... All this is too damned hard and unphysical.*"

Despite those comments being made several decades ago, and despite group theory being now a more common part in the education of physicists, significant difficulties with the standard representation theory of groups have not been overcome. There lacks the universal applicability and acceptability associated with, for example, calculus.

The first serious drawback of the standard group representation theory is that it is *unphysical*. By this I mean it was developed by mathematicians for mathematical purposes and without physical application in mind. The group was introduced into mathematics as early as 1810, and the theory of group representation was developed during the 1920's, before quantum mechanics was formulated; in this respect it is unlike calculus, which was invented about the same time as Newton's laws were discovered. Second, there is no general method for treating various kinds of group representation problems. Any given technique applies only to a particular problem and for a particular group, or class of groups. Not only do many of the methods for dealing with point groups, permutation groups, space groups, and Lie groups all differ drastically from each other, but the methods for finding the characters, irreducible basis (IRB), irreducible matrices and Clebsch–Gordan (CG) coefficients also vary from one to the other. Therefore, in many cases, these methods are more of an art than a science. Third, in physical applications, we often need to construct an IRB $\psi_m^{(\nu)}$ symmetry adapted to a given group chain $G \supset G(s)$. The standard method is to use the projection operator

$$P_{mk}^{(\nu)} = \frac{h_\nu}{g} \sum_R D_{mk}^{(\nu)}(R)^* R,$$

where $D^{(\nu)}(R)$ are irreducible matrices in the $G \supset G(s)$ IRB, which in turn depend on the $G \supset G(s)$ IRB $\psi_m^{(\nu)}$ through the relation

$$D_{mk}^{(\nu)}(R) = \langle \psi_m^{(\nu)} | R | \psi_k^{(\nu)} \rangle.$$

Now the trouble is that the $G \supset G(s)$ IRB is not known yet. Thus we are at an impasse when both the matrices and IRB are unknown.

As pointed out by Salam (1963), a battle has raged between the amateurs and professional group theorists. The amateurs have maintained that everything one needs from the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices. As an amateur myself, in this book I have introduced a new (certainly, non–professional) approach to group representation theory and it is quite interesting to note that the foundation of the new approach is precisely the theory of the complete set of commuting operators (CSCO) initiated by Dirac, the prince of amateurs in the field of group theory.

The special features of the approach are as follows.

1. *Simplicity and Accessibility*: The new representation theory for groups is essentially an extension of representation theory in quantum mechanics. Group representation theory is intimately related to quantum mechanics just as calculus is to classical mechanics. Thus it should be easily acceptable to physicists. For a group G, three kinds of CSCO are introduced, the CSCO–I, –II, and –III, roughly speaking the CSCO for the class space, the irreducible space, and the group space, respectively. They are the analogies of $\mathbf{J}^2$, $(\mathbf{J}^2, J_z)$, $(\mathbf{J}^2, J_z, J_3)$, respectively, for the rotation group in three–dimensional space.

2. *Universality and Versatility*: Based on the CSCO–I, –II, and –III, all compact (discrete or continuous) groups are treated in a unified way. Therefore once one knows the representation theory of the rotation group, in principle one knows the representation theory of all other compact groups. The new approach is constructive in nature, leading to a new method, the so–called eigenfunction method (EFM) for determining group representation. The problems of determining firstly the primitive characters and isoscalar factors, secondly the $G \supset G(s)$ IRB and CG coefficients, and thirdly the irreducible matrices, are all reduced to a single recipe: Seek the eigenvectors (or eigenfunctions) of the CSCO–I, –II, and –III of G, respectively. The EFM proves to be powerful and versatile in treating point groups, permutation groups, unitary groups, graded unitary groups and space groups, and for both vector and projective representations. The EFM for a discrete group (both for finite and infinite types) is simpler than conventional methods and is flexible enough to obtain the irreducible basis adapted to any given group chain $G \supset G(s)$ without need of any knowledge of the irreducible matrices, or conversely, to obtain all the irreducible matrices in any given $G \supset G(s)$ classification without any knowledge of the irreducible basis.

3. *Applicability*: Since the ultimate step of the method is the diagonalization of the representative matrices of a certain type of CSCO, the procedure can be easily translated into a computer program. Several standard codes are already available. Furthermore, using the eigenvalues of the CSCO as irrep labels enables us to find *algebraic solutions* for the point groups, just as one has analytic solutions for rotation group.

The book is self–contained and suitable for self–study, and is a combination of a textbook and a monograph. By ignoring some proofs and some paragraphs or passages marked with asterisks, the book becomes an easily readable textbook. The theory is developed starting with concrete examples and leading up to more abstract conclusions, as well as from the special to the general, supplemented with abundant illustrative examples. The emphasis is on the EFM technique rather than on strict rigor. Some theorems are cited without proof, since the theorems are easily understandable and their proofs can be found in many group theory textbooks. A knowledge of elementary group theory is not necessary to read this book, but a knowledge of the linear algebra and representation theory as well as angular momentum theory in quantum mechanics is assumed.

The various important coefficients, such as the Clebsch–Gordan, Racah, subduction and induction coefficients, the isoscalar factors and fractional parentage coefficients are discussed in detail for point groups, permutation groups, unitary groups and space groups. Tables for several useful coefficients are given. Some new dualities between the permutation group and unitary groups are disclosed and are fully exploited for computing many coefficients of unitary groups in terms of those of the permutation groups. The theory on roots and weights in Lie groups is also reformulated in the spirit of representation theory of quantum mechanics. The applications of group theory in quantum mechanics are discussed with emphasis on application to many-body systems. The connection between the new and standard approaches is discussed. There should be no difficulty for a reader of the present book to understand the conclusions derived in other textbooks or in the literature, although the derivations given here may be totally different.

Tables and figures are indexed according to their section numbers. For example, Table(Fig.) x.y-n denotes the n-th table (figure) in Sec. x.y. If there is only a single table or figure in the section, then the suffix "-1" will be omitted. References are indicated by the names of the first author, or first two authors, followed by the year of publication; if this is still not sufficient, then an index $[x]$ will precede the year.